Families of periodic solutions for some Hamiltonian PDEs

(**with G. Arioli**)

(1) **The problem etc.**

(2) **Main results**

(3) **Numerical results**

(4) **Proofs**

We consider **time-periodic solutions** for the
nonlinear wave equation ($\mu = 1$) and the **nonlinear beam equation** ($\mu = 2$)

$$\partial_t^2 \mathfrak{u}(t,x) + (-1)^\mu \partial_x^{2\mu} \mathfrak{u}(t,x) = f(\mathfrak{u}(t,x))\,, \qquad (t,x) \in \mathbb{R} \times (0,\pi)\,,$$

with Dirichlet BCs. These PDEs are **Hamiltonian** with

$$H(\mathfrak{u},\mathfrak{v}) = \int_0^\pi \Big[\tfrac{1}{2}(\partial_x^\mu \mathfrak{u})^2 + \tfrac{1}{2}\mathfrak{v}^2 - F(\mathfrak{u})\Big]\,dx\,, \qquad F' = f\,.$$

From a period 2π one can get "related" periods via scaling. Changes f unless homogeneous.

Our motivation:
- Observed instabilities in a bridge model [Arioli, Gazzola 2000].
- CAP for Hamiltonian and/or parabolic PDEs with potential small denominator issues.

Existing relayed work:

Variational methods for <u>period 2π</u> and related:
$\mu = 1$ [Rabinowitz 1978; Rabinowitz 1981; . . .]
$\mu = 2$ [Lee 2000; Liu 2002; Liu 2004; . . .]

Perturbative methods for <u>small $\mathfrak{u}$</u> and positive-measure sets of periods near "special" values:
$\mu = 1$ [Berti 2007; Gentile, Mastropietro, Procesi 2005; Gentile, Procesi 2009]
$\mu = 2$ [Mastropietro, Procesi 2006; Gentile, Procesi 2009]

We restrict to $f(\mathfrak{u}) = \sigma\mathfrak{u}^3$ with $\sigma = \pm 1$.
Setting $\mathfrak{u}(t,x) = u(\alpha t, x)$, where $\frac{2\pi}{\alpha}$ is the desired period for $\mathfrak{u}$, we arrive at the equation

$$\boldsymbol{L_\alpha u = \sigma u^3\,, \qquad L_\alpha = \alpha^2\partial_t^2 + (-1)^\mu \partial_x^{2\mu}\,,}$$

where $u = u(t,x)$ is 2π-periodic in t and satisfies Dirichlet boundary conditions at $x = 0, \pi$.

$\mu = 1$: Nonlinear **wave equation** $\alpha^2\partial_t^2 u - \partial_x^2 u = \sigma u^3$.

$\mu = 2$: Nonlinear **beam equation** $\alpha^2\partial_t^2 u + \partial_x^4 u = \sigma u^3$.

Consider the vector space $\mathcal{A}^{\mathrm{o}}$ of all real analytic functions

$$\boldsymbol{u = \sum_{n,k} u_{n,k} P_{n,k}\,, \qquad P_{n,k}(t,x) = \cos(nt)\sin(kx)\,.}$$

We restrict our analysis to the subspace $\mathcal{B}$ consisting of all $u \in \mathcal{A}^{\mathrm{o}}$ with the property that $u_{n,k} \neq 0$ only if n and k are both odd.
Notice that

$$L_\alpha P_{n,k} = \lambda_{n,k} P_{n,k}\,, \qquad \lambda_{n,k} = k^{2\mu} - (\alpha n)^2 = (k^\mu + \alpha n)(k^\mu - \alpha n)\,.$$

We only consider α values for which $\boldsymbol{\lambda_{n,k} \neq 0}$ for all odd n and k.

This includes the set $\mathbb{Q}_{\mathrm{o}}$ of rationals $\alpha = p/q$ with p and q of opposite parity.

Definition. *A solution $u \in \mathcal{B}$ of the equation $L_\alpha u = \sigma u^3$ will be called a type $(1,1)$ solution if $|u_{n,k}| < |u_{1,1}|$ whenever $n > 1$ or $k > 1$.*

First consider the **nonlinear wave equation** for some rational values of α. Our sample set:

$$Q_1 = \left\{\tfrac{3}{8}, \tfrac{5}{12}, \tfrac{7}{16}, \tfrac{9}{20}, \tfrac{13}{28}, \tfrac{1}{2}, \tfrac{15}{28}, \tfrac{11}{20}, \tfrac{9}{16}, \tfrac{7}{12}, \tfrac{5}{8}, \tfrac{9}{14}, \tfrac{11}{16}, \tfrac{7}{10}, \tfrac{13}{18}, \tfrac{3}{4}, \tfrac{11}{14}, \tfrac{5}{6}, \tfrac{7}{8}, \tfrac{9}{10}, \tfrac{11}{12}, \tfrac{13}{14}, \tfrac{17}{18}\right\}.$$

Theorem 1. For each $\alpha \in Q_1$ the equation $\alpha^2\partial_t^2 u - \partial_x^2 u = u^3$ has a solution $u \in \mathcal{B}$ of type $(1,1)$ with $|u_{1,1}| > \sqrt{2(1-\alpha)}$.

Remark. Every solution $u \in \mathcal{B}$ of the equation $\alpha^2\partial_t^2 u - \partial_x^2 u = u^3$ with $\alpha \in \mathbb{Q}_\mathrm{o}$ yields a solution $\tilde{u} \in \mathcal{B}$ of the equation $\alpha^2\partial_t^2\tilde{u} - \partial_x^2\tilde{u} = -\tilde{u}^3$, and vice-versa. The functions u and $\tilde{u}$ are related via $\tilde{u}(t,x) = \alpha^{-1}u(x-\pi/2, t-\pi/2)$.

Next we consider irrational values of α.

Unfortunately we have to switch to the **nonlinear beam equation**.
Still difficult to construct non-small solutions for specific α.

Best known: $\alpha = 1/\sqrt{c}$ where c is an integer that is not the square of an integer.
By Siegel's theorem on integral points on algebraic curves of genus one,

$$c\lambda_{n,k} = ck^4 - n^2 \to \infty \quad \text{as} \quad n \vee k \to \infty\,.$$

Unfortunately we have no useful bounds . . .

So we make an assumption:

Theorem 2. *Let $\alpha = 1/\sqrt{3}$. Assume that $|3k^4 - n^2| \geq 39$ for all $k \geq 9$ and all $n \in \mathbb{N}$. Then the equation $\alpha^2\partial_t^2 u + \partial_x^4 u = u^3$ has a solution $u \in \mathcal{B}$ of type $(1,1)$ with $|u_{1,1}| > 1$.*

We have verified the assumption $\min_n |3k^4 - n^2| \geq 39$ for $9 \leq k \leq 10^{12}$.

Our third result concerns irrational values of α that are close to the rationals in

$$Q_2 = \left\{\tfrac{1}{4}, \tfrac{3}{10}, \tfrac{9}{20}, \tfrac{1}{2}, \tfrac{7}{12}, \tfrac{5}{8}, \tfrac{3}{4}, \tfrac{5}{6}, \tfrac{7}{6}, \tfrac{5}{4}, \tfrac{19}{14}, \tfrac{17}{12}, \tfrac{31}{20}, \tfrac{13}{8}, \tfrac{31}{18}, \tfrac{61}{34}\right\}.$$

Theorem 3. *For each $r \in Q_2$ there exists a set $R \subset \mathbb{R}$ of positive measure that includes r as a Lebesgue density point, such that for each $\alpha \in R$, the equation $\alpha^2\partial_t^2 u + \partial_x^4 u = \sigma u^3$ with $\sigma = \operatorname{sign}(1-\alpha)$ has a solution $u \in \mathcal{B}$ of type $(1,1)$ with $|u_{1,1}| > \sqrt{2|1-\alpha|}$.*

Remark. In all of the equations considered, other types of solutions can be obtained via scaling: If $u \in \mathcal{A}^{\mathrm{o}}$ satisfies the equation $L_\alpha u = \sigma u^3$, and if we define

$$\tilde{u}(t,x) = b^{\mu} u(at, bx)\,, \qquad \tilde{\alpha} = \alpha b^{\mu}/a\,, \tag{✡}$$

with b and a nonzero integers, then $\tilde{u}$ belongs to $\mathcal{A}^{\mathrm{o}}$ and satisfies $L_{\tilde{\alpha}}\tilde{u} = \sigma\tilde{u}^3$.

In our proofs we solve $L_\alpha u = \sigma u^3$ via the fixed point equation

$$\boldsymbol{u = \mathcal{F}_\alpha(u) \stackrel{\text{def}}{=} L_\alpha^{-1}\sigma u^3\,, \qquad \sigma = \mathrm{sign}(1-\alpha)\,.}$$

For **numerical experiments** we use Fourier polynomials

$$\boldsymbol{u = \sum_{\substack{n\le N\\ k\le K}} u_{n,k}P_{n,k}\,, \qquad P_{n,k}(t,x) = \cos(nt)\sin(kx)\,.}$$

and **truncate** $\boldsymbol{u^3}$ to wavenumbers $n \le N$ and $k \le K$.
As $N \to \infty$ the equation becomes Hamiltonian, even if $K < \infty$.

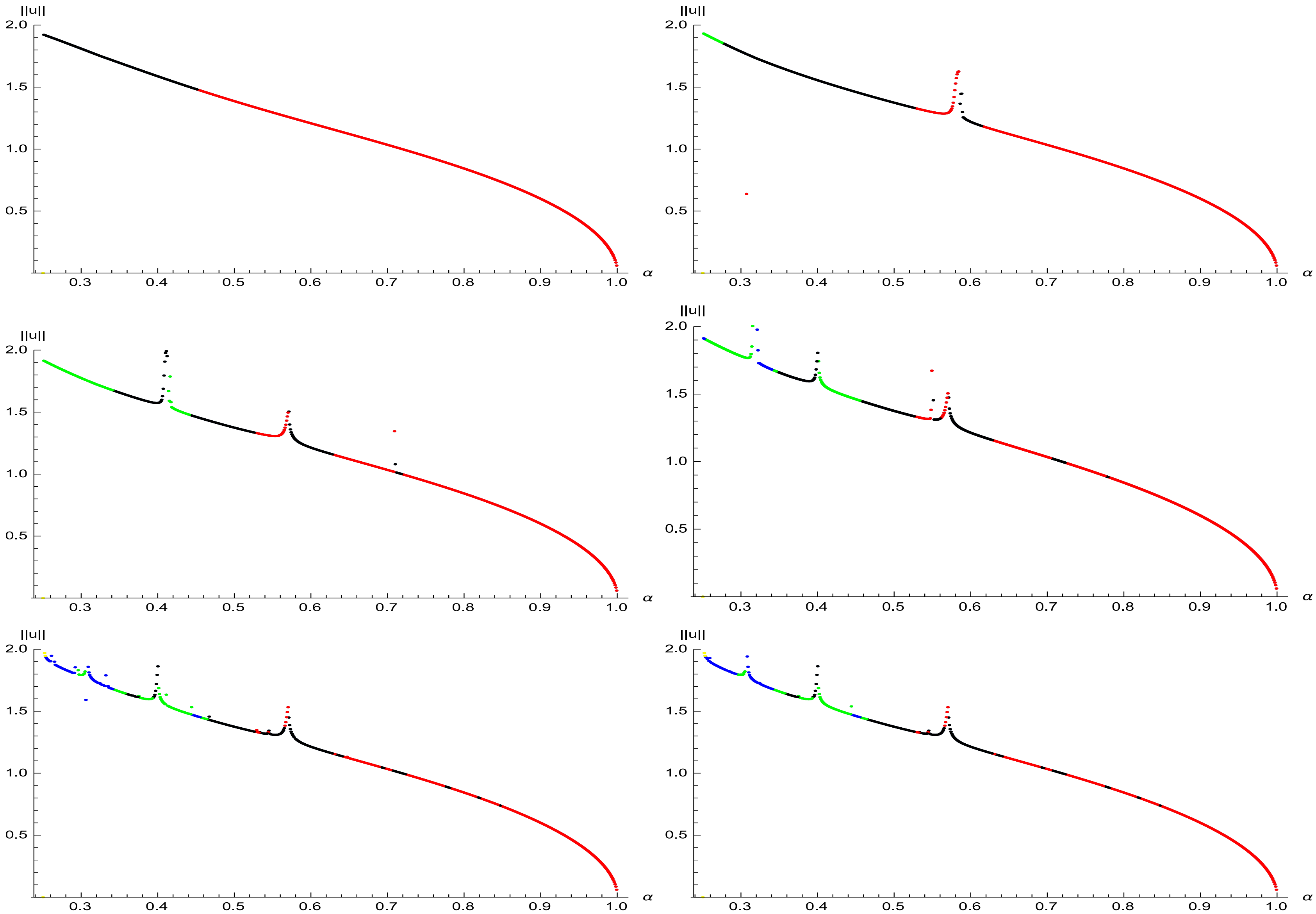
Definition for the $K < \infty$ equation. *The union of all smooth branches that include a solution of type* $(1,1)$ *will be referred to as the* $(1,1)$ *branch . Scaling each solution on the* $(1,1)$ *branch via* (✡) *yields what we will call the* (a,b) *branch .*

In the following **graphs** we show the norm

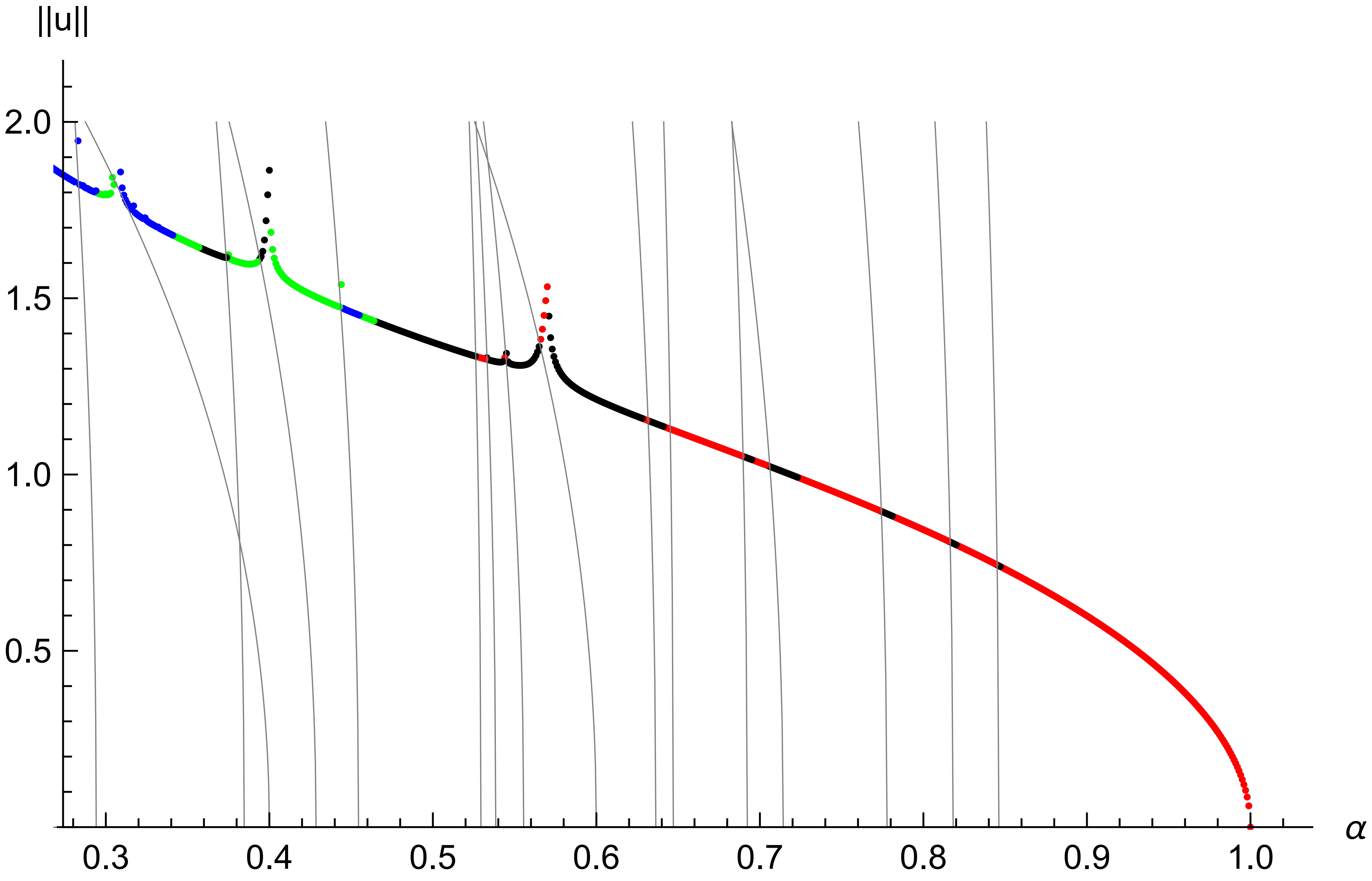
$$\boldsymbol{\|u\|_0 = \sum_{n,k} |u_{n,k}|}$$

of the numerical solution u as a function of α.
C o l o r s encode the index of u: the number of eigenvalues larger than 1 of $D\mathcal{F}_\alpha(u)$.

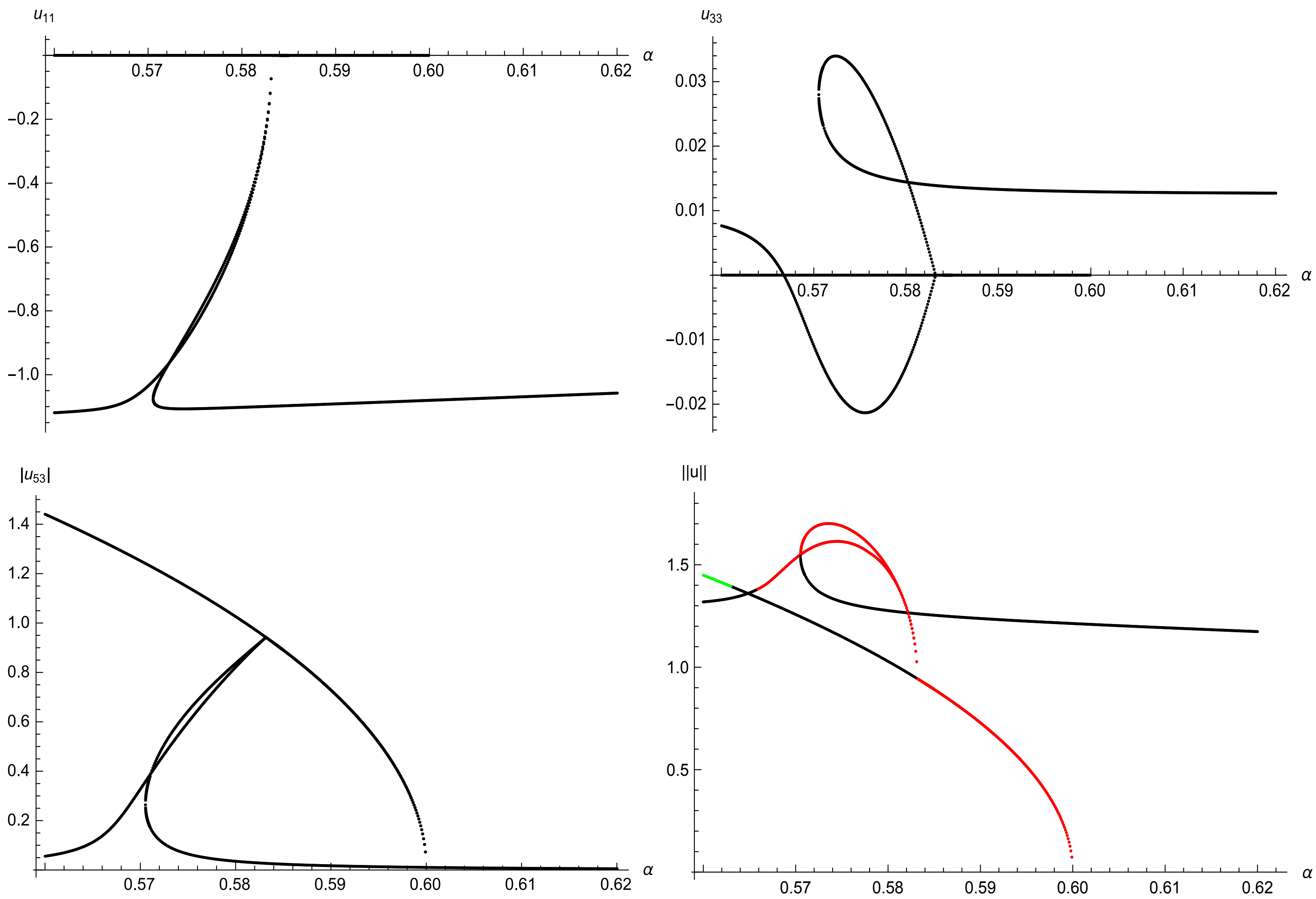
The $(1,1)$ branch for the nonlinear wave equation $\alpha^2 u_{tt} - u_{xx} = u^3$ truncated at $N = K = 3, 5, 7, 9, 19, 39$.



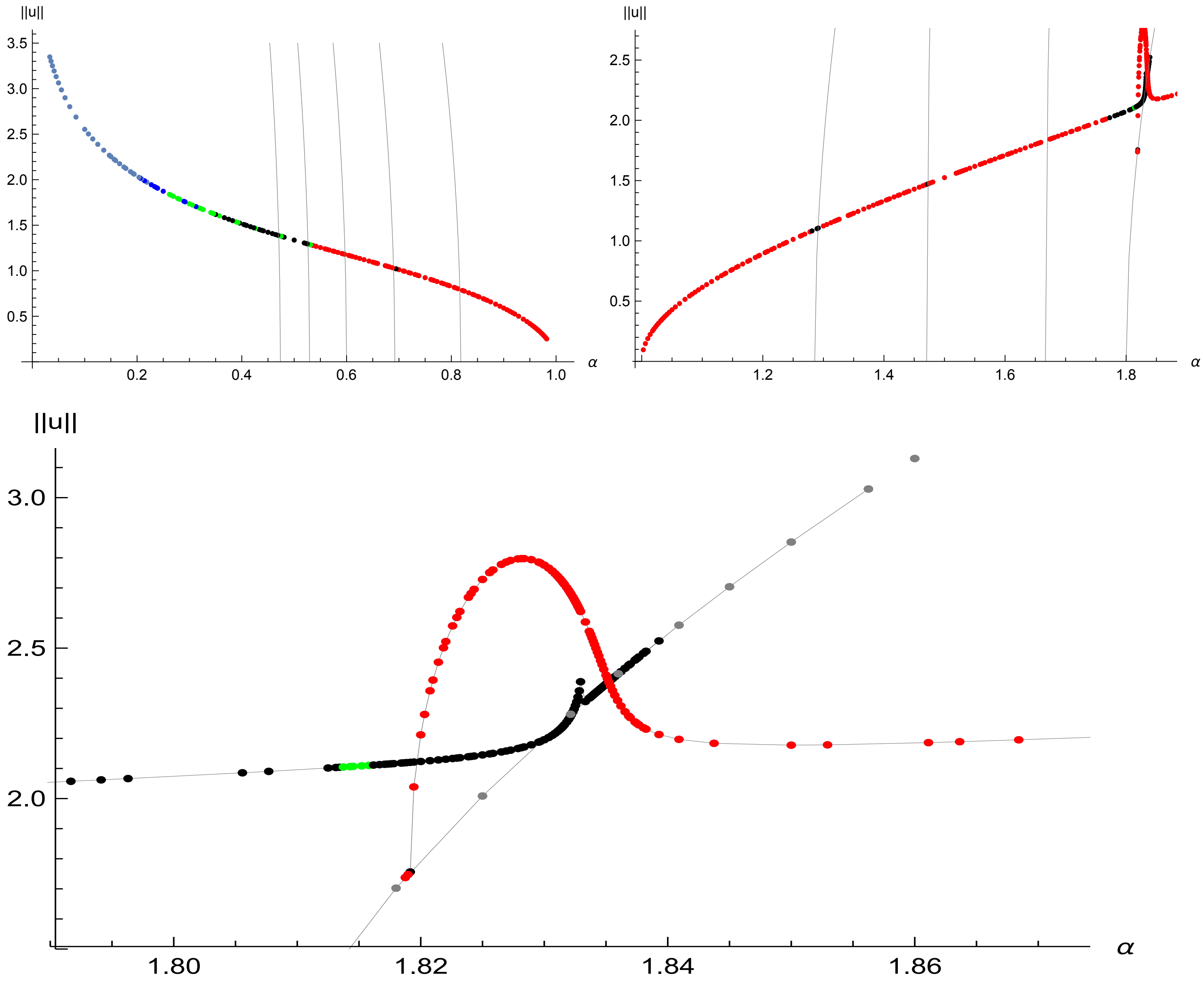
The $(1,1)$ branch for the truncated nonlinear wave equation $\alpha^2 u_{tt} - u_{xx} = u^3$ and some other (a,b) branches (thin lines).



The **nonlinear wave equation**, truncated at $N \gg K = 7$.
The $(1,1)$ branch undergoes a fold bifurcation at $\alpha \simeq 0.571$
and a pitchfork bifurcation involving the $(5,3)$ branch at $\alpha \simeq 0.585$.



The **nonlinear beam equation** $\boldsymbol{\alpha^2 u_{tt} + u_{xxxx}} = \pm \boldsymbol{u^3}$ truncated at $K = 63$ and $N = 127$.



The proofs of Theorems 1,2,3 use the contraction mapping theorem.

Given $\rho = (\rho_1, \rho_2)$ with $\rho_j > 0$ denote by $\mathcal{A}^{\circ}_{\rho}$ the closure of ... with respect to the norm

$$\|u\|_{\rho} = \sum_{n,k} |u_{n,k}| \varrho_1^n \varrho_2^k \,, \qquad \varrho_j = 1 + \rho_j \,.$$

Let $\mathcal{B}_{\rho} = \mathcal{B} \cap \mathcal{A}^{\circ}_{\rho}$. Consider a quasi-Newton map associated with $\mathcal{F}_{\alpha}$,

$$\boldsymbol{\mathcal{N}_{\alpha}(h) = \mathcal{F}_{\alpha}(u_0 + Ah) - u_0 + (\mathrm{I} - A)h} \,,$$

where u_0 is an approximate fixed point and A an approximate inverse of $\mathrm{I} - D\mathcal{F}_{\alpha}(u_0)$. Denote by B_{δ} the open ball of radius δ in $\mathcal{B}_{\rho}$, centered at the origin.

Theorem 3 is proved by verifying the following bounds.

Lemma 4 *For each $r \in Q_2$ there exists a set $R \subset \mathbb{R}$ of positive measure that includes r as a Lebesgue density point, a pair ρ of positive real numbers, a Fourier polynomial $u_0 \in \mathcal{B}_{\rho}$, a linear isomorphism $A : \mathcal{B}_{\rho} \to \mathcal{B}_{\rho}$, and positive constants K, δ, ε satisfying $\varepsilon + K\delta < \delta$, such that for every $\alpha \in R$ the map $\mathcal{N}_{\alpha}$ defined as above is analytic on B_{δ} and satisfies*

$$\|\mathcal{N}_{\alpha}(0)\|_{\rho} < \varepsilon \,, \qquad \|D\mathcal{N}_{\alpha}(h)\|_{\rho} < K \,, \quad h \in B_{\delta} \,.$$

Compactness of $L_\alpha^{-1} : \mathcal{B}_\rho \to \mathcal{B}_\rho$.

Define $\llbracket s \rrbracket = \mathrm{dist}(s, \mathbb{Z})$. A simple estimate on the eigenvalues of L_α is

$$\beta^2 |\lambda_{n,k}| = (\beta k^\mu + n)|\beta k^\mu - n| \geq \big(2(\beta k^\mu \vee n) - \llbracket \beta k^\mu \rrbracket\big)\llbracket \beta k^\mu \rrbracket\,, \qquad \beta = \alpha^{-1}\,.$$

If $\alpha = p/q$ with p odd and q even: $\llbracket \beta k^\mu \rrbracket \geq 1/p$ for k odd; so in this case L_α^{-1} is compact.

For $\mu = 2$ and irrational α we can use the following.
Let $(\psi_1, \psi_2, \psi_3, \ldots)$ be a summable sequence of nonnegative real numbers.

Proposition 5. *Let $m \geq 1$. Consider an interval J_m of length $m^{-2} \leq |J_m| \leq 1$. Then*

$$\Big\{ \beta \in J_m : \ \llbracket \beta k^2 \rrbracket \geq \psi_k \quad \text{for all} \quad k \geq m \Big\}$$

has measure at least $(1 - 4\Psi_m)|J_m|$, where $\Psi_m = \sum_{k \geq m} \psi_k$.

Applying this with $|J_m| = 1$ and $\psi_k = k^{-3/2}$ yields the

Corollary 6. *For almost every $\alpha \in \mathbb{R}$ the operator $L_\alpha = \alpha^2 \partial_t^2 + \partial_x^4$ has a compact inverse.*

Subspaces for error terms: $u \in \mathcal{B}_{\rho,\nu,\kappa}$ iff $u \in \mathcal{B}_\rho$ and $u_{n,k} = 0$ whenever $n < \nu$ or $k < \kappa$.
Our enclosures for $u \in \mathcal{B}_\rho$ consist of interval enclosures
for each $c_{n,k}$ and for the norm of each $E_{\nu,\kappa}$ in a representation

$$u = \sum_{\substack{n \le N \\ k \le K}} c_{n,k} P_{n,k} + \sum_{\substack{\nu \le 2N \\ \kappa \le 2K}} E_{\nu,\kappa}\,, \qquad E_{\nu,\kappa} \in \mathcal{B}_{\rho,\nu,\kappa}\,.$$

Estimating the map $u \mapsto u^3$ on $\mathcal{B}_\rho$ is "standard".
The operator norm of $L_\alpha^{-1} : \mathcal{B}_{\rho,\nu,\kappa} \to \mathcal{B}_{\rho,\nu,\kappa}$ is bounded by $\boldsymbol{\beta^2/\phi(\nu,\kappa)}$ where

$$\boldsymbol{\phi(\nu,\kappa) = \inf_{\substack{n \ge \nu \\ k \ge \kappa}} \beta^2 |\lambda_{n,k}| = \inf_{\substack{n \ge \nu \\ k \ge \kappa}} (\beta k^\mu + n)|\beta k^\mu - n|\,, \qquad \beta = \alpha^{-1}\,.}$$

Here ν, κ, n, k are odd positive integers. To prove Lemma 4 we use

Lemma 7. *Let $r = p/q$ with p odd and q even. Given odd positive integers κ and ν, there exists a set $R \subset \mathbb{R}$ of positive measure that includes r as a Lebesgue density point, such that for all $\alpha \in R$,*

$$\phi(\nu,\kappa) \ge \frac{7}{4p}\left[\left(\beta\kappa^2 \vee \nu\right) - \frac{7}{16p}\right], \qquad \beta = \alpha^{-1}\,.$$

Idea of the proof: In the above **inf** distinguish between $k \ge m$ and $k < m$.
For $k \ge m$ use Proposition 5 with J_m centered at r. And for $k < m$ use that α is close to r.
Do this for increasingly large m.

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